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VORTICAL FLOWS AND CANONICAL EQUATIONS OF MOTION OF A MAGNETIZABLE, PERFECTLY CONDUCTING FLUID*

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The classical Kelvin's circulation theorem and Helmholtz theory on the motion of vortex lines with the fluid and conservation of the strength of vortex tubes are generalized to the case of the vortical adiabatic flows of a magnetizable, perfectly conducting fluid. Canonical variables are found and canonical Hamiltonian equations of motion are obtained.

1. The equation of motion of the fluid in question has the form**(**Golosov V.V., Vasil'eva N.L., Taktarov N.G. and Shaposhnikova G.A. Hydrodynamic equations for polarizable, magnetizable, multicomponent and multiphase media. Discontinuous solutions. Study of discontinuous solutions with a jump in magnetic permeability. Moscow, Izd-vo MGU, 1975)

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p + \frac{B_k}{4\pi}\nabla H_k + \left[\mathbf{j}\times\frac{\mathbf{B}}{c}\right]; \tag{1.1}$$
$$p = p_0 + \frac{1}{4\pi}\int_0^H \left[\mu - \rho\left(\frac{\partial\mu}{\partial\rho}\right)_{T,H}\right]\mathbf{H}\,d\mathbf{H}$$

where $\mathbf{j}$ is the electric current density, $\mathbf{B} = \mu(\rho, T, H)\mathbf{H}$ is the magnetic induction, T is the temperature, p_0 is the pressure of the normal fluid without the magnetic field, and the remaining notation is standard. We will use below the Gibbs thermodynamic identities for a magnetizable medium

$$dU = T\,dS + \frac{p}{\rho^2}d\rho + \mathbf{H}d\frac{\mathbf{B}}{4\pi\rho}, \quad dW = T\,dS + \frac{dp}{\rho} + \mathbf{H}d\frac{\mathbf{B}}{4\pi\rho} \tag{1.2}$$
$$U = U_0 + \frac{1}{4\pi\rho}\left\{\mathbf{HB} + \int_0^H\left[T\left(\frac{\partial\mu}{\partial T}\right)_{\rho,H} - \mu\right]\mathbf{H}\,d\mathbf{H}\right.$$
$$W = W_0 + \frac{1}{4\pi\rho}\left\{\mathbf{HB} + \int_0^H\left[T\left(\frac{\partial\mu}{\partial T}\right)_{\rho,H} - \rho\left(\frac{\partial\mu}{\partial\rho}\right)_{T,H}\right]\mathbf{H}\,d\mathbf{H}\right.$$
$$S = S_0 + \frac{1}{4\pi\rho}\int_0^H\left(\frac{\partial\mu}{\partial T}\right)_{\rho,H}\mathbf{H}\,d\mathbf{H}$$

Here U, W, S denote, respectively, the internal energy, enthalpy and entropy per unit mass of the fluid, and the zero subscripts denote the parameters without a magnetic field.

Using relations (1.2) we can write (1.1) in the form

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$$\frac{\partial \mathbf{v}}{\partial t} - [\mathbf{v} \times \operatorname{rot} \mathbf{v}] = T\, dS - \nabla\left(\frac{v^2}{2} + W - \frac{\mathbf{HB}}{4\pi\rho}\right) + \frac{1}{c\rho}[\mathbf{j} \times \mathbf{B}] \tag{1.3}$$

Let us take the curl of (1.3), remembering that $\mathbf{j} = (c/4\pi) \operatorname{rot} \mathbf{H}$. Introducing now the operator $\operatorname{Helm} \boldsymbol{\omega} \equiv d\boldsymbol{\omega}/dt - (\boldsymbol{\omega}\nabla)\, \mathbf{v} + \boldsymbol{\omega} \operatorname{div} \mathbf{v}$, $\boldsymbol{\omega} \equiv \operatorname{rot} \mathbf{v}$, we can reduce the result to the form

$$\operatorname{Helm} \boldsymbol{\omega} = \nabla T \times \nabla S - \operatorname{rot}\left[\frac{\mathbf{B}}{4\pi\rho} \times \operatorname{rot} \mathbf{H}\right] \tag{1.4}$$

According to Friedman's theorem /1/ the necessary and sufficient condition for the Helmholtz theorems on the motion of vortex lines with the fluid and conservation of the strength of vortex tubes to be satisfied is the condition $\operatorname{Helm} \boldsymbol{\omega} = 0$, i.e. according to (1.4), the condition

$$\nabla T \times \nabla S = \operatorname{rot}\left[\frac{\mathbf{B}}{4\pi\rho} \times \operatorname{rot} \mathbf{H}\right] \tag{1.5}$$

On the other hand, we know that the rate of change of the circulation Γ of the velocity vector $\mathbf{v}$ over a closed contour C, moving together with the fluid, is equal to /2/

$$\frac{d\Gamma}{dt} = \frac{d}{dt} \oint_C \mathbf{v}\, d\mathbf{r} = \frac{d}{dt} \iint_\sigma \operatorname{rot} \mathbf{v}\, d\boldsymbol{\sigma} = \iint_\sigma \operatorname{Helm} \boldsymbol{\omega}\, d\boldsymbol{\sigma}$$

where $d\mathbf{r}$ is the element of the tangent to C and σ denotes the surface bounded by the contour C. Substituting into it the expression $\operatorname{Helm} \boldsymbol{\omega}$ from (1.4) we find, that in order to preserve the circulation of velocity over the closed contour (Kelvin's theorem) when the fluid is in continuous motion, condition (1.5) is necessary and sufficient, just as in the Helmholtz theorems. In the special case of ideal magnetohydrodynamics we obtain from (1.5), when $\mu = \text{const} = 1$, the well-known analogous condition /3/.

Condition (1.5) represents a very strong constraint. We can also show here that a generalized vorticity exists for the medium for which the vorticity theorems considered here hold without the strict condition (1.5). We will show this blow, using the variational formulation of the problem.

2. The variational formulation of the problem concerning the motions in question of the magnetizable conducting fluid, was given in /4/. Let us modify it slightly.

We take the density l of the Lagrangian in the form

$$l = \frac{\rho \mathbf{v}^2}{2} - \rho U + \varphi\left(\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v}\right) - \alpha\left(\frac{\partial S}{\partial t} + \mathbf{v}\cdot\nabla S\right) - \beta\left(\frac{\partial a}{\partial t} + \mathbf{v}\cdot\nabla a\right) + \gamma \operatorname{div} \mathbf{B} + \mathbf{b}\left[\frac{\partial \mathbf{B}}{\partial t} - \operatorname{rot}(\mathbf{v} \times \mathbf{B})\right]$$

where a is an arbitrary Lagrangian coordinate of the fluid particle and $\varphi, \alpha, \beta, \gamma, \mathbf{b}$ are the Lagrange multipliers. Carrying out the identity transformation and neglecting the divergent terms and terms containing the partial time derivatives, we obtain

$$l_1 = \frac{\rho \mathbf{v}^2}{2} - \rho U - \rho \frac{\partial \varphi}{\partial t} + S \frac{\partial \alpha}{\partial t} + a \frac{\partial \beta}{\partial t} - \mathbf{B}\left(\frac{\partial \mathbf{b}}{\partial t} + \nabla \gamma\right) - \mathbf{v}\,(\rho \nabla \varphi + \alpha \nabla S + \beta \nabla a + \mathbf{B} \times \operatorname{rot} \mathbf{b}) \tag{2.1}$$

Taking the variational derivatives in l_1 over the free functions $\mathbf{v}, \rho, S, a, \mathbf{B}/(4\pi\rho)$ and equating them to zero, we obtain the conditions of stationarity, which represent the Lagrangian equations of motion

$$\frac{\delta l_1}{\delta \mathbf{v}} = 0, \quad \mathbf{v} = \nabla \varphi + \frac{\alpha}{\rho} \nabla S + \frac{\beta}{\rho} \nabla a + \left[\frac{\mathbf{B}}{\rho} \times \operatorname{rot} \mathbf{b}\right] \tag{2.2}$$

$$\frac{\delta l_1}{\delta \rho} = 0, \quad \frac{v^2}{2} - W - \frac{d\varphi}{dt} + \frac{\mathbf{BH}}{4\pi\rho} = 0 \tag{2.3}$$

$$\frac{\delta l_1}{\delta S} = 0, \quad \frac{d}{dt}\left(\frac{\alpha}{\rho}\right) = T \tag{2.4}$$

$$\frac{\delta l_1}{\delta a} = 0, \quad \frac{d}{dt}\left(\frac{\beta}{\rho}\right) = 0 \tag{2.5}$$

$$\frac{\delta l_1}{\delta (\mathbf{B}/4\pi\rho)} = 0, \quad \frac{\mathbf{H}}{4\pi} = [\mathbf{v} \times \operatorname{rot} \mathbf{b}] - \left(\frac{\partial \mathbf{b}}{\partial t} + \nabla \gamma\right) \tag{2.6}$$

It can be shown that we can reduce the equations (2.2)–(2.6) to the equations of motion (1.1) by eliminating the Lagrange multipliers $\varphi, \alpha, \beta, \gamma, \mathbf{b}$.

Let us now introduce, with help of the generalized Klebsch transformation (2.2), the generalized velocity $\mathbf{u}$ and vorticity $\boldsymbol{\Omega}$

$$\mathbf{u} = \mathbf{v} - \frac{\alpha}{\rho}\nabla S - \left[\frac{\mathbf{B}}{\rho} \times \operatorname{rot}\mathbf{b}\right], \quad \mathbf{\Omega} = \operatorname{rot}\mathbf{u} \tag{2.7}$$

Using (2.7), the induction equation in the form $d(\mathbf{B}/\rho)/dt = (\rho)^{-1}\mathbf{B}\mathbf{v}\cdot\nabla)\,\mathbf{v}$ and the vector identity for three arbitrary vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

$$\mathbf{a}_1 \times \operatorname{rot}(\mathbf{a}_2 \times \mathbf{a}_3) + \mathbf{a}_2 \times \operatorname{rot}(\mathbf{a}_3 \times \mathbf{a}_1) + \mathbf{a}_3 \times \operatorname{rot}(\mathbf{a}_1 \times \mathbf{a}_2) =$$
$$\operatorname{grad}\{\mathbf{a}_1[\mathbf{a}_2 \times \mathbf{a}_3]\} + (\mathbf{a}_1 \times \mathbf{a}_2)\operatorname{div}\mathbf{a}_3 + (\mathbf{a}_2 \times \mathbf{a}_3)\operatorname{div}\mathbf{a}_1 + (\mathbf{a}_3 \times \mathbf{a}_1)\operatorname{div}\mathbf{a}_2$$

we reduce the equations of motion (1.1) to the quasibarotropic form

$$\frac{\partial \mathbf{u}}{\partial t} - [\mathbf{v} \times \mathbf{\Omega}] + \nabla\left\{\frac{v^2}{2} + W + \frac{\alpha}{\rho}\frac{\partial S}{\partial t} + [\mathbf{v} \times \operatorname{rot}\mathbf{b}]\frac{\mathbf{B}}{\rho} - \frac{\mathbf{BH}}{4\pi\rho}\right\} = 0 \tag{2.8}$$

Taking the curl of (2.8), we obtain $\operatorname{Helm}\mathbf{\Omega} = 0$. This means that the Kelvin and Helmholtz theorems considered here hold for the generalized vorticity $\mathbf{\Omega}$ without the strict condition (1.5). The same result was obtained in standard hydrodynamics by a different method in /5, 6, 7/.

3. Let us find the canonical Hamiltonian equations of motion of a magnetizable, perfectly conducting compressible fluid. To do this we transform l_1 (2.1) thus

$$l_1 = \frac{\rho \mathbf{v}^2}{2} - \rho U - \frac{\partial(\rho\varphi)}{\partial t} + \varphi\frac{\partial\rho}{\partial t} + S\frac{\partial\alpha}{\partial t} + a\frac{\partial\beta}{\partial t} -$$
$$\rho\mathbf{v}\left(\nabla\varphi + \frac{\alpha}{\rho}\nabla S + \frac{\beta}{\rho}\nabla a + \frac{\mathbf{B}}{\rho} \times \operatorname{rot}\mathbf{b}\right) + \mathbf{b}\frac{\partial\mathbf{B}}{\partial t} -$$
$$\frac{\partial(\mathbf{bB})}{\partial t} + \gamma\operatorname{div}\mathbf{B} - \operatorname{div}(\gamma\mathbf{B})$$

Using (2.2) here and neglecting terms containing time derivatives and the divergent term, we arrive at the expression

$$l_2 = -\frac{1}{2\rho}(\rho\nabla\varphi + \alpha\nabla S + \beta\nabla a + \mathbf{B} \times \operatorname{rot}\mathbf{b})^2 - \rho U + \varphi\frac{\partial\rho}{\partial t} + S\frac{\partial\alpha}{\partial t} + a\frac{\partial\beta}{\partial t} + \mathbf{b}\frac{\partial\mathbf{B}}{\partial t}$$

This yields the densities of generalized impulses determined by the formula $\pi_k = \partial l_2/\partial q_k^{\cdot}$, where $q_k = \rho, \alpha, \beta, \mathbf{B}$. Thus we obtain

$$\pi_1 = \frac{\partial l_2}{\partial \rho^{\cdot}} = \varphi, \quad \pi_2 = \frac{\partial l_2}{\partial \alpha^{\cdot}} = S, \quad \pi_3 = \frac{\partial l_2}{\partial \beta^{\cdot}} = a, \quad \pi_4 = \frac{\partial l_2}{\partial \mathbf{B}^{\cdot}} = \mathbf{b}$$

The density h of the Hamiltonian if given by the formula

$$h = \sum \pi_k \frac{\partial q_k}{\partial t} - l_2 = \frac{1}{2\rho}(\rho\nabla\varphi + \alpha\nabla S + \beta\nabla a + \mathbf{B} \times \operatorname{rot}\mathbf{b})^2 + \rho U$$

i.e. it is the total energy density of the fluid.

According to classical theory /8/ the canonical equations of motion of a continuum are given in the form $\partial q_k/\partial t = \delta h/\delta\pi_k$, $\partial\pi_k/\partial t = -\delta h/\delta q_k$. Therefore we obtain the following canonical equations for the fluid in question:

$$\frac{\partial\rho}{\partial t} = \frac{\delta h}{\delta\varphi} = -\operatorname{div}(\rho\mathbf{v}), \quad \frac{\partial\varphi}{\partial t} = -\frac{\delta h}{\delta\rho} = \frac{v^2}{2} - \mathbf{v}\cdot\nabla\varphi - W + \frac{\mathbf{BH}}{4\pi\rho} \tag{3.1}$$

$$\frac{\partial\alpha}{\partial t} = \frac{\delta h}{\delta S} = \rho T - \operatorname{div}(\alpha\mathbf{v}), \quad \frac{\partial S}{\partial t} = -\frac{\delta h}{\delta\alpha} = -\mathbf{v}\cdot\nabla S \tag{3.2}$$

$$\frac{\partial\beta}{\partial t} = \frac{\delta h}{\delta a} = -\operatorname{div}(\beta\mathbf{v}), \quad \frac{\partial a}{\partial t} = -\frac{\delta h}{\delta\beta} = -\mathbf{v}\cdot\nabla a \tag{3.3}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \frac{\delta h}{\delta\mathbf{b}} = \operatorname{rot}[\mathbf{v} \times \mathbf{B}], \quad \frac{\partial\mathbf{b}}{\partial t} = -\frac{\delta h}{\delta\mathbf{B}} = [\mathbf{v} \times \operatorname{rot}\mathbf{b}] - \left(\nabla\gamma + \frac{\mathbf{H}}{4\pi}\right) \tag{3.4}$$

We see that (3.1) corresponds to the equation of continuity and Eq.(2.3), which is in fact the Lagrange-Cauchy integral. Eqs.(3.2) are equivalent to (2.4) and the condition of adiabaticity (3.3) corresponds to (2.5) and condition of conservation of the Lagrangian coordinate of the fluid particle. Finally, (3.4) are equivalent to the induction equation and (2.6).

We note that in order to find the variational derivatives $\delta h/\delta\mathbf{b}, \delta h/\delta\mathbf{B}$, we write h in the form $h = \mathbf{v}(\rho\nabla\varphi + \alpha\nabla S + \beta\nabla a + \mathbf{B} \times \operatorname{rot}\mathbf{b}) - \rho\mathbf{v}^2/2 + \rho U$. Moreover, when calculating $\delta h/\delta\mathbf{B}$, we add to h the zero term $(-\gamma\operatorname{div}\mathbf{B})$ and the divergent term $\operatorname{div}(\gamma\mathbf{B})$, which is allowed.

The resulting Hamiltonian formalism is useful for the fluid in question in many ways. The introduction of canonical variables makes it possible to establish certain general rules governing the wave interaction in non-linear media, to derive truncated equations describing, with various degrees of approximation, simplified models of non-linear media, and pass in a natural manner to the relativistic and quantum theory generalizations /8/. In addition, the canonical Hamiltonian formulation of hydrodynamic problems is found to be convenient in the case of numerical calculations /9/.

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SLOW MOTION OF A PARTICLE IN A WEAKLY ANISOTROPIC VISCOUS FLUID*

V.N. POKROVSKII and A.A. TSKHAI

The problem of the steady flow past a rigid sphere of a linear, homogeneous weakly anisotropic viscous incompressible fluid is studied in the Stokes approximation. The solution is sought using the perturbation method and has the form of an expansion in particular solutions of the Laplace equation in Cartesian coordinates. Expressions for the velocity and pressure fields in the fluid are obtained, as well as for the force acting on the particle.

When studying certain systems such as liquid crystals, we encounter the problem of determining the coefficients of resistance when a particle is in translational and rotational motion through an anisotropic fluid. The simplest case of such a fluid is a linear, homogeneous, viscous anisotropic liquid defines by the equation (see e.g. /1/)

$$\sigma_{ij} = -p\delta_{ij} + \eta_{ijhq}\nabla_q v_h \qquad (1)$$
$$(\eta_{ijhq} = \eta_{jihq} = \eta_{ijqh} = \eta_{hqij},\ \nabla_q = \partial/\partial x_q)$$

where σ_{ij} is the stress tensor, p is the pressure, v_h is the velocity and η_{ijhq} is the tensor of viscosity coefficients with the idicated symmetry properties.

We can separate from the tensor of viscosity coefficients η_{ijhq} a part corresponding to an isotropic fluid with viscosity coefficient η

$$\eta_{ijhq} = \eta(\delta_{ih}\delta_{jq} + \delta_{iq}\delta_{jh}) + \xi_{ijhq} \qquad (2)$$

Henceforth we shall regard the anisotropic term ξ_{ijhq} as small, and this will make it possible to express the particle resistance coefficients in the form of an expansion in terms of the small parameter ξ_{ijhq}. We will restrict ourselves to determining the first-order correction to the resistance coefficient of a spherical particle in translational motion.

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